A simplified theory of magnetohydrodynamic isotropic turbulence

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The simplest case of turbulent motion in a conductive fluid is studied. The turbulence is assumed incompressible, isotropic, homogeneous, charge invariant and free of fourth-order cumulants. The emphasis is placed on certain integrals of the correlation functions such as kinetic and magnetic energy, vorticity, and current. A system of non-linear ordinary differential equations is derived which governs these integral quantities. Several cases are solved numerically, illustrating the decay of ordinary turbulence, the buildup of magnetic energy by a linear or a non-linear process, the buildup of kinetic energy, as well as the destruction of vorticity by Lorentz forces.

In order to handle certain dissipative effects, a special hypothesis is introduced which seems to promote mathematical simplicity. In particular, it leads to a simple decay law very similar to the decay law of ordinary turbulence.

1. Introduction

The ordinary kind of turbulence involves only two forms of energy; kinetic energy and heat. If the fluid is electrically conductive, a third form of energy can appear: magnetic energy. Essentially, fluid particles may behave as electric motors and generators, and fluctuations of the electric current and of the magnetic field can develop. The energy will decay into heat either by viscous friction or by ohmic losses. This is called magnetohydrodynamic (MHD) turbulence.

An interesting property of MHD turbulence is that energy can be transferred reversibly from the kinetic form to the magnetic form. This transfer is related to the work of the Lorentz forces; it can produce considerable magnetic fluctuations and thereby increase the production of heat. Another interesting property comes from the fact that the Lorentz forces, proportional to the cross-product of electric current and magnetic field, are not conservative. Therefore, they can either increase of decrease the angular momentum of a fluid particle. This results in the production or destruction of vorticity, by an isotropic and homogeneous mechanism.

The main objective of this paper is to study the mechanisms for energy exchange and for vorticity production from the point of view of classical fluid dynamics. Numerous assumptions are necessary in order to progress toward this objective, and they are listed carefully. This leads to a system of equations describing the gross features of a simplified form of MHD turbulence. This system can be integrated by an electronic computer, and several numerical solutions are given and discussed.

2. Related work and outlines

Magnetohydrodynamic turbulence is inherently far more complicated than ordinary turbulence. When the magnetic fluctuations are sufficiently small, they are governed by the velocity fluctuations and do not react on the flow. Isotropic turbulence has been the object of several contributions, notably by Batchelor (1950), Batchelor & Proudman (1956) and Moffatt (1961). The case of interacting kinetic and magnetic fluctuations has been studied by Tatsumi (1960), Roberts & Tatsumi (1960), and Chadrasekhar (1955). Chandrasekhar discussed a variety of correlation functions of the second and third order which can be built up from kinetic and magnetic fluctuations and wrote down the equivalent of the Karman–Howarth equation. Tatsumi focused his attention on the two energy spectra, and, by discarding the fourth-order cumulants, obtained a closed but very complicated system of integral equations. As for anisotropic forms of MHD turbulence, the possibility of an eddy resistivity bearing some analogy to Reynolds stresses has been pointed out by Kovasznay (1960).

In general, it appears that formal analysis of the simplest forms of MHD turbulence leads to impassable roads. In view of the meagre results obtained by those who have studied the correlation functions of ordinary turbulence, the hopes for a successful analysis of MHD correlations are very dim. Let us remember that, in the case of ordinary turbulence, one of the simplest questions is not yet satisfactorily answered. Indeed, the rate of production of mean square vorticity has repeatedly been measured with an observed non-dimensional value of $0{\cdot}4 \pm 0{\cdot}05$. The theory of Proudman & Reid (1954) cannot yield numerical values unless the viscous terms are dropped. In the non-viscous treatment, their theory predicts an asymptotic value 0·78 with the implication that the mean square vorticity becomes infinite. According to Betchov (1956), the same quantity cannot exceed the limit 0·756. The argument of the latter paper is purely kinematic and does not invoke the dynamic equations. Both theories assume that the fourth-order cumulants are negligible and this may be the cause of the poor agreement with the experimental value.

In this paper, we abandon the idea of dealing with correlation functions and attempt to describe the turbulence by certain key quantities, few enough to be handled by an electronic computer. In the case of ordinary turbulence, there are three key quantities, and these are the mean kinetic energy (integral of the spectrum), the mean square vorticity (second moment of the spectrum), and the skewness (second moment of the spectral transfer function). The time derivatives of these three quantities can be expressed from basic equations and they depend upon some additional quantities such as the fourth moment of the spectrum. In order to evaluate these additional quantities, one must make some assumption about the shape of the spectrum or the effect of phase relations. This amounts to postulating the general shape of certain correlation functions, without determining certain over-all scales. The result is a closed set of three non-linear ordinary differential equations which describes the gross properties of ordinary isotropic, homogeneous, and incompressible turbulence.

The same procedure can be applied to MHD turbulence for which the number of key quantities increases from three to eight. Several other quantities must be determined by some assumption, and for the time being, we have chosen a simple and general type of relation, which can always be reconsidered at some later stage. In particular, the recent work of Moffatt (1961) has given some clues on possible spectral shapes.

3. Basic equations and notations

Notations

x_i, t	space and time co-ordinates
u_i	velocity
h_i	magnetic field in kinematic units (that is, multiplied by $\sqrt{(\mu/\rho)}$)
μ, ρ	permeability and density, both taken as unity
ν	kinematic viscosity
λ	kinematic resistivity (with conductivity σ, $\lambda = 1/\mu\sigma$)
p, P	kinematic pressure, and kinematic total pressure.

Double correlations

K, M	twice the kinetic and twice the magnetic energy per unit mass
R	mean square vorticity
J	mean square electric current
l_1, l_2	characteristic fine scales of R and J (see equations (5.5))

Triple correlations

S	rate of vorticity production (see equation (5.6))
X	rate of energy transfer (see equation (5.8))
Ω	correlation between vorticity and curl of Lorentz forces (see equations (5.10) and (5.9))
Z, Y, Λ	defined by equations (5.10) and (5.9)
D_S, D_X, D_Y, D_Ω	groups of dissipative terms associated with the changes of S, X, Y, Ω

Conventions

$u_{ikj} = \partial^2 u_i / \partial x_k \partial x_j$

$p_i = \partial p / \partial x_i$

$a_i b_i = \sum_1^3 a_i b_i$

$\langle\ \rangle$	ensemble average or space average
c_X, c_S, c_Ω	constants used in expressions for D_X, D_S, D_Ω
α, β, γ	constants used in expressions for l_1, l_2 (see equations (7.4) and (7.6))

Basic equations

The fluid is assumed incompressible, viscous, and electrically conducting. It is convenient to take the density as unity and to measure the magnetic field in units such that the magnetic energy density is simply $\frac{1}{2}h_i h_i$. This amounts to representing the magnetic field by the corresponding Alfvén velocity. The equations of motion are

$$\frac{\partial u_i}{\partial t} + u_k u_{ik} - \nu u_{ikk} = -p_i + h_k h_{ik} - h_k h_{ki}, \tag{3.1}$$

$$\frac{\partial h_i}{\partial t} + u_k h_{ik} - \lambda h_{ikk} = h_k u_{ik}, \tag{3.2}$$

$$u_{kk} = h_{kk} = 0. \tag{3.3}$$

The total pressure P is defined as

$$P = p + \tfrac{1}{2} h_k h_k, \tag{3.4}$$

and from equation (3.1), it satisfies the relation

$$P_{jj} = -u_{pq} u_{qp} + h_{pq} h_{qp}. \tag{3.5}$$

4. Basic assumptions

We shall make the following six assumptions.

(i) and (ii) The turbulence is homogeneous and isotropic. This restricts the problem to a simple situation without violating any physical law.

(iii) The turbulence is incompressible. This is in violation of what we can expect in a plasma, but it greatly simplifies the analysis.

(iv) The turbulence is charge invariant. This means that the turbulent properties are assumed invariant under a transformation that reverses the sign of all electric charges. Since the basic equations are charge invariant, this assumption only requires that the initial conditions and boundary conditions be charge invariant. For example, this assumption implies that the correlation between fluid velocity and electric current, both taken at the same point, must vanish. Indeed there is no *a priori* reason why one kind of electric charge would accompany the fluid rather than another. This assumption would break down if the microstructure of the plasma should become important. It eliminates many terms and is perhaps not necessary.

(v) All fourth-order cumulants are zero. This is the well-known hypothesis of Milliontchikoff, which permits reduction of quadruple correlations to products of double correlations. This assumption is needed to obtain a closed system of equations.

(vi) The dissipative effects have a simple damping action. This novel assumption is described in detail in § 7. Since we do not use the full correlation functions, an assumption of this kind is necessary to close the system of equations.

5. A catalogue of tensors and invariants

In this paper, we are concerned only with mean products of quantities measured at the same point. Beginning with the velocity, we use the assumptions of homogeneity, isotropy, and incompressibility to show that

$$\langle u_i u_j \rangle = \tfrac{1}{3} K \delta_{ij}, \tag{5.1}$$

$$\langle u_{ij} u_{kl} \rangle = \frac{R}{30}(4\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}), \tag{5.2}$$

where K is twice the kinetic energy per unit mass, and R is the mean square vorticity. In particular,

$$K = \langle u_i u_i \rangle, \quad R = \langle u_{ij} u_{ij} \rangle, \quad \langle u_{ij} u_{ji} \rangle = 0. \tag{5.3}$$

Similarly, the corresponding relations for magnetic field correlations are

$$M = \langle h_i h_i \rangle, \quad J = \langle h_{ij} h_{ij} \rangle, \quad \langle h_{ij} h_{ji} \rangle = 0, \tag{5.4}$$

where M is twice the mean magnetic energy and J is the mean square electric current.
We shall also encounter mean square second derivatives, which will be expressed in terms of the characteristic lengths l_1 and l_2 defined by the following relations:

$$\langle u_{ijj} u_{ikk} \rangle = R/l_1^2, \quad \langle h_{ijj} h_{ikk} \rangle = J/l_2^2. \tag{5.5}$$

Invariants such as $\langle u_i u_{ijjkk} \rangle$ can easily be expressed in terms of $\langle u_{ijj} u_{ikk} \rangle$ by assuming homogeneity and incompressibility.

The only triple velocity correlation encountered in this paper is defined as

$$S = -\langle u_{ij} u_{jk} u_{ik} \rangle. \tag{5.6}$$

It indicates the rate of stretching of the vorticity. The following relation can also be demonstrated

$$\langle u_{ij} u_{jk} u_{ki} \rangle = 0. \tag{5.7}$$

The magnetic field is involved in other triple products, such as

$$X = \langle u_{ij} h_i h_j \rangle, \tag{5.8}$$

which measures the power generated by the Lorentz forces or the stretching of the magnetic field. It is simply related to invariants such as $\langle u_i h_j h_{ij} \rangle$. Out of a total of 22 invariants containing one velocity component, two magnetic field components, and three derivatives, such as $\langle u_{ijkk} h_i h_j \rangle$, or $\langle u_i h_{ji} h_{jkk} \rangle$, only four are independent (see appendix). We define

$$\left.\begin{aligned} E &= \langle u_{ij} h_{ik} h_{jk} \rangle, \quad F = \langle u_{ij} h_{ik} h_{kj} \rangle, \\ G &= \langle u_{ij} h_{ki} h_{kj} \rangle, \quad H = -\langle u_{ij} h_j h_{ikk} \rangle, \end{aligned}\right\} \tag{5.9}$$

and also

$$\Omega = E + F - H, \quad Z = H - G, \quad Y = F - E - G, \quad \Lambda = E + G. \tag{5.10}$$

It can be shown (see Appendix) that Ω is the mean product of the vorticity and the curl of the Lorentz forces, and that Y measures the stretching of the electric current by the rate of strain. The quantity Z will be identified later with the rate of production of electric currents.

The quadruple products of velocity components and their derivatives can be reduced to double correlations by assuming that all fourth-order cumulants are zero. If two derivatives are involved, this leads to terms proportional to KR. If four derivatives are involved, this leads to terms either in R^2 or in KRl_1^{-2}. Since l_1 is usually the smallest scale of the motion, the terms in KRl_1^{-2} are larger than the terms in R^2. Similar expressions exist for the quadruple products of magnetic fields, leading to terms in MJl_2^{-2} and J^2. Again, we surmize that the terms in MJl_2^{-2} will be larger than the terms in J^2. We also encounter mixed velocity-magnetic invariants leading to terms in MR, MRl_1^{-2}, or RJ.

The pressure appears in many terms and a general discussion would be quite lengthy. Since the pressure terms do not play an essential role in the final equations, we shall only summarize our results. Considering invariants such as $\langle P_{ij}u_iu_j\rangle$, one finds they are all of the order of KR, multiplied by numerical factors Ψ_1 or Ψ_2 that depend upon the shape of the kinetic energy spectrum. Invariants such as $\langle P_{ij}h_ih_j\rangle$ are of the order of MJ with factors Ψ_3 and Ψ_4 that depend upon the shape of the magnetic energy spectrum. Invariants such as $\langle P_{ij}u_{ik}u_{jk}\rangle$ contain four derivatives, and can be expressed in terms of six invariants by using isotropy, homogeneity, and incompressibility. After considerations of the pressure equation, only three independent quantities remain, and we have the following useful relations

$$\left.\begin{aligned}
&\langle P_{ij}u_{ik}u_{jk}\rangle = -(\Phi_2+\tfrac{16}{15})R^2/8, \quad \langle P_{ii}u_{jk}u_{jk}\rangle = R^2/15,\\
&\langle P_{ij}u_{ki}u_{kj}\rangle = (\Phi_2+\tfrac{24}{15})R^2/8, \quad \langle P_{ii}u_{jk}u_{kj}\rangle = -4R^2/15,\\
&\langle P_{ij}u_{ik}u_{kj}\rangle = -2R^2/15, \quad \langle Pu_iu_{ijjkk}\rangle = -(\Phi_1+\Phi_2+\tfrac{4}{3})R^2/8.
\end{aligned}\right\} \tag{5.11}$$

The factors Φ_1 and Φ_2 are of the order of unity and depend upon the shape of the kinetic energy spectrum. With $a = 2kk'/(k^2+k'^2)$, we have

$$\left.\begin{matrix}\Phi_1\\ \\ \Phi_2\end{matrix}\right\} = \frac{1}{R^2}\iint(k^2+k'^2)E(k)E(k')\begin{Bmatrix}\dfrac{1-a^2}{a^3}\ln\dfrac{1-a}{1+a}+\dfrac{2}{a^2}-\dfrac{4}{3}\\ \\ \dfrac{(1-a^2)^2}{a^3}\ln\dfrac{1-a}{1+a}+\dfrac{2}{a^2}-\dfrac{10}{3}\end{Bmatrix}dk\,dk'. \tag{5.12}$$

Invariants such as $\langle P_{ij}h_{ik}h_{jk}\rangle$ are given by relations similar to equations (5.11) except that R^2 must be replaced by $-J^2$. The sign reversal comes from the pressure relation in equations (3–5). The magnetic energy spectrum must be used to calculate two new coefficients, Φ_3 and Φ_4.

During the calculations of quadruple products, one often encounters quantities such as $\langle u_{ij}u_{kl}\rangle\langle u_{ij}u_{lk}\rangle$, which must be evaluated according to equation (5.2). Each bracket contains two indices indicating components and two indices indicating derivatives. If the two component indices of the first bracket are summed with either the two component indices or the two derivative indices of the second bracket, the result is $\frac{4}{30}R^2$. If the component indices of the first bracket are summed with one component index and one derivative index of the second bracket, the result is $-\frac{1}{30}R^2$. If summation occurs within a bracket, between a component index and a derivative index, the result is zero. Otherwise the result is $\frac{1}{3}R^2$. Thus, for example, we have

$$\left.\begin{aligned}
&\langle u_{ij}u_{pq}\rangle\langle u_{ij}u_{pq}\rangle = \tfrac{4}{30}R^2, \quad \langle u_{ij}u_{pq}\rangle\langle u_{qi}u_{jp}\rangle = \tfrac{4}{30}R^2,\\
&\langle u_{ij}u_{pq}\rangle\langle u_{ij}u_{qp}\rangle = -\tfrac{1}{30}R^2, \quad \langle u_{ij}u_{jp}\rangle\langle u_{iq}u_{pq}\rangle = 0,\\
&\qquad\langle u_{ij}u_{pj}\rangle\langle u_{qi}u_{qp}\rangle = \tfrac{1}{3}R^2.
\end{aligned}\right\} \tag{5.13}$$

6. The exact dynamic equations

A number of important relations can be derived from the basic equations by applying suitable operators and averaging. These equations read as follows:

$$\frac{1}{2}\frac{dK}{dt} = -X - \nu R, \tag{6.1}$$

$$\frac{1}{2}\frac{dM}{dt} = X - \lambda J, \tag{6.2}$$

$$\frac{dX}{dt} = \frac{M}{3}(R-J) - \Psi_3 \frac{MJ}{12} - D_X, \tag{6.3}$$

$$\frac{1}{2}\frac{dR}{dt} = S + \Omega - \nu \frac{R}{l_1^2}, \tag{6.4}$$

$$\frac{1}{2}\frac{dJ}{dt} = Z - \lambda \frac{J}{l_2^2}, \tag{6.5}$$

$$\frac{dS}{dt} = \tfrac{1}{6}R^2 - D_S, \tag{6.6}$$

$$\frac{d\Omega}{dt} = -\frac{M}{3}\left(\frac{R}{l_1^2} - \frac{J}{l_2^2}\right) - D_\Omega, \tag{6.7}$$

$$\frac{dZ}{dt} = \frac{M}{3}\left(\frac{R}{l_1^2} - \frac{J}{l_2^2}\right) + \tfrac{4}{5}RJ - \left(\frac{1}{5} + \frac{\Phi_4}{4}\right)J^2 - D_Z, \tag{6.8}$$

$$\frac{dY}{dt} = \tfrac{1}{5}J^2 - D_Y, \tag{6.9}$$

$$\frac{d\Lambda}{dt} = -D_\Lambda. \tag{6.10}$$

Equations (6.1) and (6.2) show that energy is exchanged between the kinetic and magnetic modes at the rate X. The friction losses are expressed by νR and the ohmic losses by λJ.

In the equation for the rate of change of X, all the dissipative terms have been included in a single term defined as

$$D_X = -\nu \langle u_{ijkk} h_i h_j \rangle - \lambda \langle u_{ij} h_{ikk} h_j \rangle - \lambda \langle u_{ij} h_i h_{jkk} \rangle. \tag{6.11}$$

Furthermore, the term in Ψ_3 is of minor importance, and we can say that the nonlinear build up of X is controlled by the difference $R-J$. Thus, if $R > J$, X tends to become positive and to deplete the kinetic mode. This tends to reduce R. If $R < J$, energy is fed to the kinetic mode and R tends to grow. This indicates a tendency to distribute the energy in such a way that R and J will be equal.

The vorticity equation (equation (6.4)) contains, in addition to the ordinary terms, the quantity Ω that represents the production or destruction of vorticity by the Lorentz forces. The rate of change of Ω is controlled according to equation (6.7) in which we have grouped nine dissipative terms in the expression

$$D_\Omega = -\nu \langle u_{ijmm} h_{ik} h_{jk} \rangle - \dots - \dots - \lambda \langle u_{ij} h_j h_{ikkmm} \rangle. \tag{6.12}$$

The rate of change of S is controlled by equation (6.6), which does not contain any magnetic term, and where D_S includes all the dissipative terms.

The currents are controlled by equation (6.5) where Z appears as a production term. The variations of Z are given by equation (6.8), which resembles equation (6.7) except for the presence of special terms in RJ and in J^2. These terms will turn out to be of negligible importance, and in general Z is not too different from $-\Omega$. Since Y does not appear in any other equation, equation (6.9) will not be mentioned again in this paper. Finally, equation (6.10) indicates that Λ would be invariant in the absence of dissipative processes. The terms D_Z, D_Y, and D_Λ collect various dissipative terms and play the same roles as D_S and D_Ω.

7. Evaluation of the dissipative terms and closure

We shall now introduce the sixth basic assumption, which is specially designed to express the length l_1 and l_2 and all the D quantities in terms of convenient quantities. This reduces the number of unknowns and leads to a closed system.

Ordinary turbulence

The eddy Reynolds number of an ordinary turbulent flow can be defined as

$$\mathscr{R}_\nu = \frac{K}{\nu R^{\frac{1}{2}}}. \tag{7.1}$$

When this number is low (say, less than 10), the energy spectrum is not very different from a Gaussian spectrum and l_1^2 is proportional to K/R. For a Gaussian spectrum, one can show that

$$l_1^2 = \frac{K}{1{\cdot}4R}. \tag{7.2}$$

When the Reynolds number is large, the spectrum falls according to Kolmogoroff's law and cuts off abruptly above a certain wave-number. This wave-number corresponds to the length l_1, and we have

$$l_1^2 = \frac{\nu}{\alpha R^{\frac{1}{2}}}, \tag{7.3}$$

where α is a universal constant.

In order to have an expression valid for large and small Reynolds numbers, we shall assume the following relation, which combines equations (7.2) and (7.3)

$$\frac{1}{l_1^2} = 1{\cdot}4\frac{R}{K} + \alpha\frac{R^{\frac{1}{2}}}{\nu}. \tag{7.4}$$

A quantity such as S or D_S depends no only upon the energy spectrum but also upon the phase relations between triplets of Fourier components. One can surmize that S cannot exceed some limit proportional to $R^{\frac{3}{2}}$ (see Betchov 1956) or that D_S cannot exceed some limit proportional to R^2. These considerations have led us to assume the relation

$$D_S = c_S R^{\frac{1}{2}} S, \tag{7.5}$$

where c_S is a constant. Since D_S is obtained by applying the operator $\nu(\partial^2/\partial x_i \partial x_i)$ to the various parts of S, this assumption replaces the Laplace operator by the factor l_1^{-2}.

MHD Turbulence

We shall now follow the clues from ordinary turbulence, and postulate the relation

$$\frac{1}{l_2^2} = 1{\cdot}4\frac{J}{M}+\beta\frac{J^{\frac{1}{2}}}{\lambda}+\gamma\frac{R^{\frac{1}{2}}}{\lambda}, \tag{7.6}$$

where the constants β and γ permit the assumption of different types of cut-off. We shall also assume the relation

$$D_X = c_X(R^{\frac{1}{2}}+2J^{\frac{1}{2}})\,X \tag{7.7}$$

with similar expressions for D_Ω and D_Z. If $R > J$, this amounts to replacing the viscous diffusion operator by $R^{\frac{1}{2}}$. However, if $R < J$, it amounts to replacing the resistive diffusion operator by $J^{\frac{1}{2}}$. The factor 2 corresponds to the presence of two magnetic terms in equation (6.11).

A closed system

By combining the exact dynamic equations and the assumed expressions for the dissipative terms, one can now obtain a closed system of equations. We programmed an electronic computer to determine the development of K, M, X, R, J, S, Ω, and Z by using equations (6.1) to (6.8) supplemented by equations (7.4) to (7.7). We specified the initial values of K, M, R, and J and always assumed zero for the initial values of the triple correlations X, S, Ω, and Z. We also specified arbitrary values for the numerical constants, as indicated with each example.

In order to avoid large numbers, we used the meter and the millisecond as fundamental units. Thus, a hydrogen plasma of 10^{17} protons/cm^3 having fluctuations of 10^3 G r.m.s. will have a magnetic energy per unit mass of $M = 10^8\,\mathrm{m^2/sec^2}$. In our calculations, this corresponds to $M = 10^2\,\mathrm{m^2/msec^2}$. If the characteristic length $(M/J)^{\frac{1}{2}}$ is 1 cm, we have $J = 10^6\,\mathrm{msec^{-2}}$. The same plasma, at a temperature of 10^5 °K will have a viscosity of $10^{-2}\,\mathrm{m^2/sec}$, which we convert to $\nu = 10^{-5}$ m^2/msec (Linhardt 1960). The conductivity can be estimated at 10^3 mho/cm and the magnetic viscosity becomes $10\,\mathrm{m^2/sec}$, which we convert to $\lambda = 10^{-2}$ m^2/msec. These particular values are useful as references for the interpretation of figures 3 to 8.

8. A simplified system of equations

After solving numerically the system formed by the exact equations with the expressions assumed for the dissipative terms, we realized that certain simplifications were permissible. Since the coefficients Ψ_3 in equation (6.3) is of the order of unity, the term in which it appears can be neglected without serious consequences. It simply means that the production of X vanishes at $R = J$ and not at some slightly larger values of R. The terms in RJ and J^2 in equation (6.8) are negligible in comparison with the other terms, and can be dropped (Φ_4 is also of the order of unity). Since we always assumed $c_Z = c_\Omega$ and took identical initial values for Z and Ω (namely, $Z = \Omega = 0$ at $t = 0$), this last step leads to the results $Z = -\Omega$.

We shall also drop the term $2J^{\frac{1}{2}}$ in the expression for D_X and D_Ω since the choice of the constants c_X and c_Ω is of far greater importance. Finally, we shall drop the term giving l_1 at low Reynolds numbers, retain only the term with the constant γ in the expression (7.6) for l_2, and assume $\alpha = \gamma$. With these simplifications, the system becomes

$$\frac{1}{2}\frac{dK}{dt} = -X - \nu R, \tag{8.1}$$

$$\frac{1}{2}\frac{dM}{dt} = X - \lambda J, \tag{8.2}$$

$$\frac{1}{2}\frac{dR}{dt} = S + \Omega - \alpha R^{\frac{3}{2}}, \tag{8.3}$$

$$\frac{1}{2}\frac{dJ}{dt} = -\Omega - \alpha R^{\frac{1}{2}} J, \tag{8.4}$$

$$\frac{dX}{dt} = \frac{M}{3}(R - J) - c_X R^{\frac{1}{2}} X, \tag{8.5}$$

$$\frac{dS}{dt} = \frac{R^2}{6} - c_S R^{\frac{1}{2}} S, \tag{8.6}$$

$$\frac{d\Omega}{dt} = -\frac{\alpha M R^{\frac{1}{2}}}{3}\left(\frac{R}{\nu} - \frac{J}{\lambda}\right) - c_\Omega R^{\frac{1}{2}}\Omega. \tag{8.7}$$

This system has a simple decaying solution in which K and M decay as t^{-1}; R, J, and X as t^{-2}; S and Ω at t^{-3}. Since the Reynolds number stays constant during such decay, the system formed by the exact equations and the assumed dissipative terms has the same property.

The system shows clearly that it is the difference $R - J$ which controls the energy transfer, with a time lag controlled by c_X. Since $R^{-\frac{1}{2}}$ has the dimension of a time, it plays the role of a variable time constant during the evolution of the turbulence. The quantity Ω is controlled by the difference $R/\nu - J/\lambda$. In general, λ is larger than ν and Ω tends to become negative. In such cases, the fluid behaves as a homopolar generator and we shall define the homopolar régime as the state of turbulence when Ω exceeds $-S$. This means that the Lorentz forces decrease the mean square vorticity faster than the usual stretching process can operate. The close relation between $-\Omega$ and Z is perhaps related to the conservation of a canonical angular momentum in particle electrodynamics.

9. Ordinary turbulence

For a discussion of ordinary turbulence, we drop all the magnetic terms. This gives the following system

$$\frac{1}{2}\frac{dK}{dt} = -\nu R, \tag{9.1}$$

$$\frac{1}{2}\frac{dR}{dt} = S - \alpha R^{\frac{3}{2}}, \tag{9.2}$$

$$\frac{dS}{dt} = \tfrac{1}{6}R^2 - c_S R^{\frac{1}{2}} S. \tag{9.3}$$

For low Reynolds numbers, equation (9.2) must be modified since the first term on the right-hand side of equation (7.4) must be used in place of the second. This leads to a decay law with K proportional to $t^{-2.5}$, as expected for a Gaussian spectrum.

Since equations (9.2) and (9.3) form a closed system, a simple analysis is possible. By eliminating S, one obtains a second-order equation in R, where

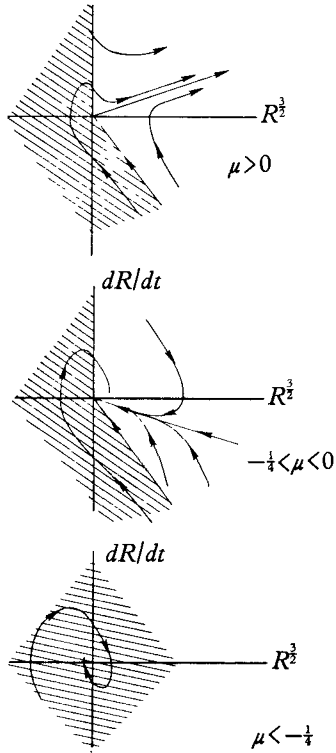

FIGURE 1. Evolution of the mean square vorticity R in ordinary turbulence, for various values of the parameter μ. In the shaded region, the solution takes unacceptable negative values.

the solutions can be outlined in the plane formed by dR/dt and $R^{\frac{3}{2}}$. The only important parameter μ is defined as

$$\mu = \frac{\frac{1}{2} - 3\alpha c_S}{(3\alpha + c_S)^2}. \tag{9.4}$$

As seen in figure 1, several solutions lead to unacceptable negative values of R. It is only if $-\frac{1}{4} < \mu < 0$ that the solution decays properly. With these equations and given initial conditions, there is one particular initial value $K = K_e$ such that $K(\infty)$ vanishes. If $K(0) > K_e$, K would run negative, according to (9.1). However, the Gaussian term of (7.4) giving l_1 would no longer be negligible. If this term is retained, one finds that, as K approaches zero, the dissipation of R is increased and K decays without sign reversal. If $K(0) > K_e$, R may vanish while K approaches a constant value. This also means that $\mathscr{R}_\nu$ tends to infinity. Physically this means that some large eddies persist with a negligible rate of energy dissipation.

Clearly this situation should be unstable and the theory should provide for some effect of K on the stability of the $R-S$ system. It is not clear whether K should affect the dissipative term D_S or whether K should appear in the expression of a fourth order cumulant. In particular, a deviation from the assumption of zero cumulants could introduce a term in KR/l_1^2 in equation (9.3) for dS/dt. In order to clarify this point, it would be very helpful to have experimental evidence on the variation of $S/R^{\frac{3}{2}}$ with Reynolds number.

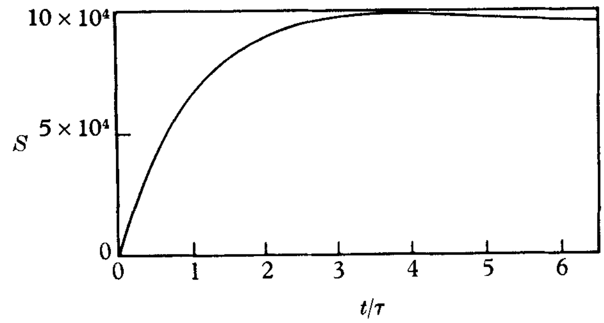

FIGURE 2. The rapid build-up of the triple correlation in ordinary turbulence. $\nu = 10^{-4}$; $\alpha = 0{\cdot}1$; $R(0) = 10^4$; $K(0) = 1$; $\mathscr{R}_\nu = 100$; $c_S = 3{\cdot}5$; $\tau = 0{\cdot}0033$.

From known experimental values of S and R we have estimated $\alpha = 0{\cdot}12$ and c_S near 3. Starting from $S = 0$, one finds that S grows rapidly and then decays as t^{-3}, as shown in figure 2. The characteristic time of the build-up is

$$\tau = \frac{1}{c_S} R^{-\frac{1}{2}}. \tag{9.5}$$

For $0 < t/\tau < 2$, the viscous effects are negligible and our solution agrees with that of Proudman & Reid (1954). For $t/\tau > 5$, the dissipation closely matches the production and we have dynamic equilibrium. Thus our solution contains both the initial and the final phases.

10. Some numerical calculations

Several calculations have been performed with the full set of equations, including the terms in RJ and J^2 and the Gaussian contributions to l_1 and l_2. In general, the triple correlations build up very rapidly from zero in a time of the order of τ (see equation (9·5)). These calculations do not correspond to specific problems; they are simply designed to illustrate various processes.

A decaying magnetic field is shown in figure 3. Although J initially increases, the ohmic losses lead to an exponential decay.

The exponential growth of the magnetic model shown in figure 4 is due to the effect of M on the build-up of X. A small initial value of M, together with a large R leads to an increase in M. Eventually the transfer uses up all the kinetic energy.

The opposite case is shown in figure 5. Beginning with small values of K and R, we observe a build-up of K. Except for the early rise, R decreases because $-\Omega$ exceeds S: we have a homopolar régime.

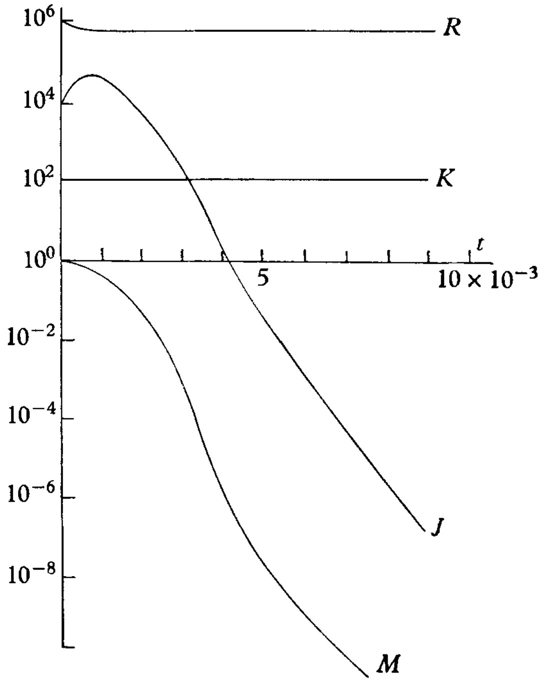

FIGURE 3. An example of decaying magnetic energy M. The kinetic energy K remains constant. $\nu = 10^{-4}$; $\lambda = 10^{-2}$; $\alpha = 0{\cdot}1$; $\beta = 0{\cdot}1$; $\gamma = 0$; $c_X = c_\Omega = c_Z = c_S = 4$.

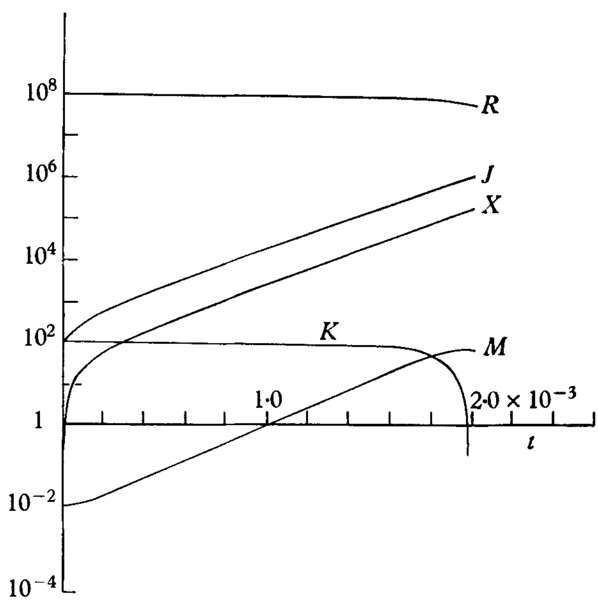

FIGURE 4. An example of magnetic instability. The magnetic energy M, the mean square electric current J and the magnetic power X grow exponentially. $\nu = 10^{-5}$, $\lambda = 10^{-2}$; $\alpha = 10^{-1}$; $\gamma = 1$; $\beta = 0$; $c_X = \frac{1}{2}$; $c_Z = c_\Omega = 10$; $c_S = 4$.

Figure 6 shows another typical case of homopolar régime with slight production of K. The ratio $-\Omega/S$ is of the order of 3 during the process.

In figure 7 we examine the role of the friction constants. If the dissipative terms are too small, negative and meaningless values of R can be obtained. However, if the constants are sufficiently large, the solutions are well-behaved.

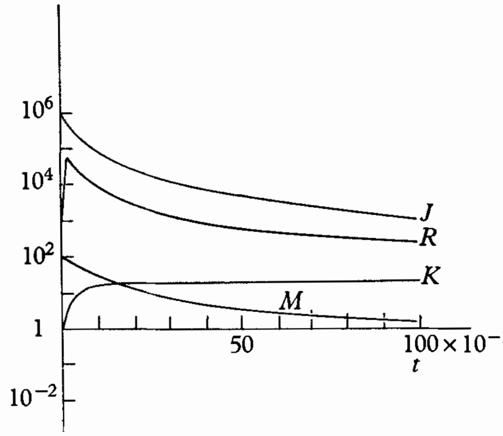

FIGURE 5. Build-up of the kinetic energy K at the expense of the magnetic energy M. $\nu = 10^{-4}$; $\lambda = 10^{-2}$; $\alpha = 0{\cdot}1$; $\beta = 0{\cdot}1$; $\gamma = 0$; $c_X = c_\Omega = c_Z = c_S = 4$.

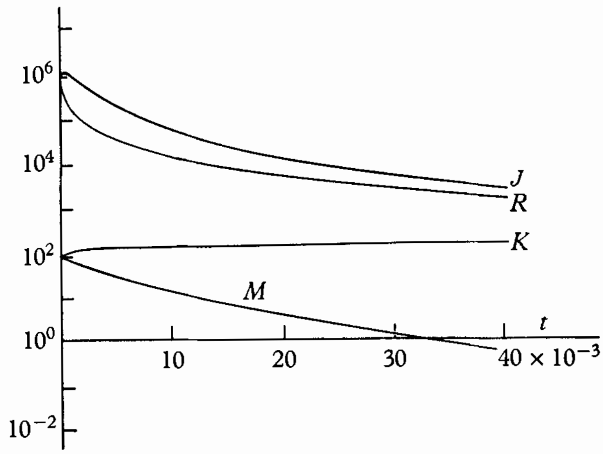

FIGURE 6. An example of strong interaction, where the destruction of vorticity by the Lorentz forces exceeds the turbulent production. $\nu = 10^{-4}$; $\lambda = 10^{-2}$; $\alpha = 0{\cdot}1$; $\beta = 0{\cdot}1$; $\gamma = 0$; $c_X = c_\Omega = c_Z = c_S = 4$; $Y > S$.

Since the ohmic losses are linear in J, while the transfer of energy is nonlinear, one can expect that the initial value of M may be of decisive importance. An example of this process is shown in figure 8, which shows the results of three calculations starting from identical initial conditions except for the changes in $M(0)$. For $M(0) = 3$, the magnetic mode decays from $t = 10^{-4}$ msec on. For $M(0) = 10$, it first drops, but the transfer becomes significant near $t = 10^{-3}$. Finally, M and J are stabilized at some slowly decaying levels.

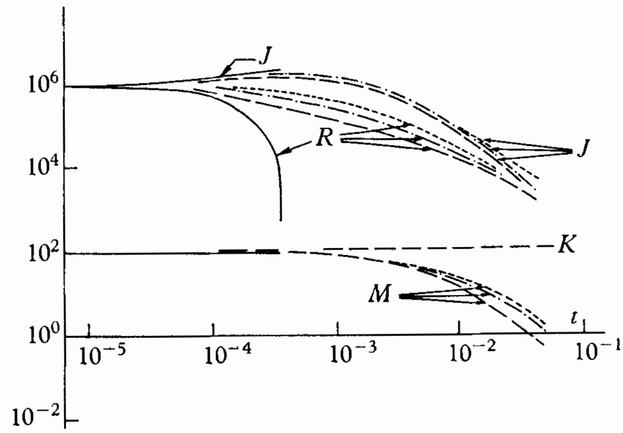

FIGURE 7. Analysis of the role of the constants in the case of figure 6. $\nu = 10^{-4}$; $\lambda = 10^{-2}$; $\alpha = \beta = 0{\cdot}1$; $\gamma = 0$; $c_X = c_\Omega = c_Z = c_S = c$ (say); ——, $c = 2$; ----, $c = 4$; —·—·, $c = 8$; ····, $c = 16$.

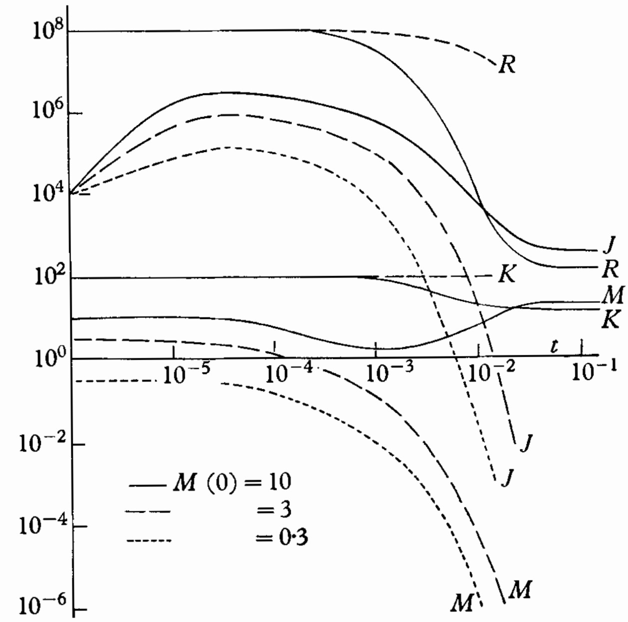

FIGURE 8. Non-linear dynamo build-up of M. These three cases have the same initial conditions except for $M(0)$. If $M(0)$ is sufficiently large, a non-linear build-up occurs. $\nu = 10^{-5}$; $\lambda = 10^{-2}$; $\alpha = \beta = 0{\cdot}1$; $\gamma = 0$; $c_X = 0{\cdot}5$; $c_\Omega = c_Z = 16$; $c_S = 4$. —— $M(0) = 10$; ---- $M(0) = 3$; ···· $M(0) = 0{\cdot}3$.

11. The case of weak Lorentz forces

When the magnetic variables are sufficiently small, that is, when $M \ll K$, $J \ll R$, the velocity field modifies the magnetic field while the magnetic field no longer affects the velocity field. This is equivalent to the statement that the Lorentz forces are negligible.

Time dependence

Let us consider the case where the kinematic averaged quantities are stationary (or nearly so). The basic equations now depend linearly upon the magnetic field, and we can assume without loss of generality that the averaged magnetic quantities vary as $\exp(2qR^{\frac{1}{2}}t)$. This leads to the equations

$$qR^{\frac{1}{2}}M = X - \lambda J, \tag{11.1}$$

$$(q+\gamma)R^{\frac{1}{2}}J = -\Omega, \tag{11.2}$$

$$(2q+c_X)X = \tfrac{1}{3}MR^{\frac{1}{2}}, \tag{11.3}$$

$$(2q+c_\Omega)\,\Omega = -\frac{\alpha MR}{3\nu}, \tag{11.4}$$

$$l_2^{-2} = \frac{\gamma R^{\frac{1}{2}}}{\lambda}, \tag{11.5}$$

where we have not assumed $\alpha = \gamma$, as previously.

A discussion of this fourth-order system shows that there is at least one unstable solution if

$$\frac{\alpha\lambda c_X}{\gamma\nu c_\Omega} < 1. \tag{11.6}$$

If this inequality is not satisfied, two positive roots are still possible. Note that c_Ω has a destabilizing effect because it leads to low currents and low ohmic losses.

Stationary flow

We shall now take $q = 0$ and assume that the spectrum of h_i has a maximum near some wave-number k_2 and then drops as k^{-n} up to the Kolmogoroff cut-off at $k = k_1$. We shall also assume that $2 < n < 4$ so that J is determined by the components near k_2 while l_2 is comparable to l_1, with $k_1 l_1 \approx 1$. This means that

$$J \approx Mk_2^2. \tag{11.7}$$

We shall now estimate the highest possible value of X. Since it depends upon two spectral components of h_i and one of u_{ij}, it is clear that X cannot exceed some ceiling proportional to $MR_L^{\frac{1}{2}}$ where R_L is the contribution to mean-square vorticity R from components with wave-numbers between zero and $2k_2$. With the assumption that the velocity spectrum falls as $k^{-\frac{5}{3}}$ (Kolmogoroff), we find

$$R_L = R\left(\frac{2k_2}{k_1}\right)^{\frac{4}{3}}. \tag{11.8}$$

We can now define a non-dimensional rate of energy transfer as

$$\xi = \frac{X}{2^{\frac{2}{3}}MR^{\frac{1}{2}}}\left(\frac{k_1}{k_2}\right)^{\frac{2}{3}}, \tag{11.9}$$

and expect $|\xi|$ to remain smaller than some number of order unity.

Let us now consider Ω. It depends upon E, F, and H (see equation (5.10a)). Looking at equations (5.9), we note that H contains higher derivatives than E

or F and therefore should be the dominant term. With the assumed shape of the magnetic energy spectrum and referring to equation (5.5*b*), one finds

$$k_2 l_2 = (k_2/k_1)^{\frac{1}{2}(5-n)}. \qquad (11.10)$$

If $k_2 l_2 < 1$, H is the dominant term in equation (5.10*a*) and the maximum value of Ω can be estimated at

$$\Omega^2 < RM^2 k_2^2 l_2^{-2}. \qquad (11.11)$$

We now define a non-dimensional rate of change of the currents as

$$\zeta = \frac{-\Omega l_2}{R^{\frac{1}{2}} M k_2} \qquad (11.12)$$

and we expect $|\zeta|$ to remain smaller than some number of order unity.

For a stationary flow, it follows from the energy relation, equations (11.1) and (11.9), that

$$k_2 = \sqrt{2}\, \xi^{\frac{3}{4}} R^{\frac{3}{8}} \lambda^{-\frac{3}{4}} k_1^{-\frac{1}{2}}. \qquad (11.13)$$

Since the rate of kinetic energy dissipation is given by

$$\epsilon = \nu R = \nu^3 k_1^4, \qquad (11.14)$$

this corresponds to

$$k_2 = \sqrt{2}\, \xi^{\frac{3}{4}} (\epsilon/\lambda^3)^{\frac{1}{4}}. \qquad (11.15)$$

This corresponds to a relation given by Moffatt (1961) without the factor ξ.

By introducing equation (11.12) into the current relation (equation (11.2)), we obtain

$$\gamma k_2 l_2 = \zeta. \qquad (11.16)$$

We can eliminate γ and l_2 from this expression by using equations (11.5) and (11.10). The result, using (11.14), is

$$\frac{k_2}{k_1} = \left(\frac{\zeta \nu}{\lambda}\right)^{\frac{2}{n-1}}. \qquad (11.17)$$

We have learned from ordinary turbulence that S reaches about half its maximum value, estimated on the assumption of zero fourth order cumulants. This leads to the idea that ξ and ζ should also be nearly constant and of the order of unity. For this to be true, equations (11.15) and (11.17) must coincide and this is possible only if $n = 11/3$. This particular spectral law has already been found by Golitsyn (1960) and by Moffatt (1961) by different procedures.

We can also use equations (11.3), (11.4), and (11.5) to obtain the damping constants, with the results

$$\gamma = 2^{\frac{1}{3}} \xi^{\frac{1}{2}} \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}}, \qquad (11.18)$$

$$c_X = \frac{1}{6\xi^{\frac{3}{2}}} \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}}, \qquad (11.19)$$

$$c_\Omega = \frac{\alpha}{6 \,.\, 2^{\frac{1}{3}} \xi^2} \left(\frac{\lambda}{\nu}\right). \qquad (11.20)$$

This means that the dissipation of Ω is primarily due to the terms in νk_1^2 while the dissipation of J and X is equivalent to an operator $(\nu\lambda)^{\frac{1}{2}} k_1^2$. This is due to the fact that the spectra of X and J are not as rich in high wave-numbers as the spectrum of Ω. If we define the wave-number $k_3 = k_2^{\frac{1}{3}} k_1^{\frac{2}{3}}$, we can express D_X as

$$D_X = \lambda k_3^2 X. \tag{11.21}$$

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REFERENCES

Batchelor, G. K. 1950 *Proc. Roy. Soc.* A, **201**, 406.
Batchelor, G. K. & Proudman, I. 1956 *Phil. Trans.* A, **248**, 46.
Betchov, R. 1956 *J. Fluid Mech.* **1**, 497.
Chandrasekhar, S. 1955 *Proc. Roy. Soc.* A, **233**, 322.
Golitsyn, G. S. 1960 *Soviet Phys. Doklady*, **5**, 536.
Kovasznay, L. S. G. 1960 *Rev. Mod. Phys.* **32**, 815.
Linhardt, J. G. 1960 *Plasma Physics*. Amsterdam: North Holland Publishing Co.
Moffatt, K. 1961 *J. Fluid Mech.* **7**, 625.
Proudman, I. & Reid, W. H. 1954 *Proc. Roy. Soc.* A, **247**, 163.
Roberts, P. H. & Tatsumi, T. 1960 *J. Math. Mech.* **9**, 697.
Tatsumi, T. 1960 *Rev. Mod. Phys.* **32**, 807.

Appendix

In order to study the invariants containing one component of u, two components of h, and three derivatives, we begin by listing all relevant combinations (the second equality will be demonstrated later):

$$
\begin{aligned}
a &= \langle u_i h_j h_{ijkk} \rangle = H, & j &= \langle u_{ij} h_k h_{ijk} \rangle = E-H,\\
b &= \langle u_j h_i h_{ijkk} \rangle = G, & k &= \langle u_{ji} h_j h_{ikk} \rangle = F-E,\\
c &= \langle u_i h_{jk} h_{ijk} \rangle = -E, & l &= \langle u_{ji} h_k h_{ijk} \rangle = 0,\\
d &= \langle u_j h_{ij} h_{ikk} \rangle = -G, & m &= \langle u_{jk} h_i h_{ijk} \rangle = -G,\\
e &= \langle u_j h_{ik} h_{ijk} \rangle = 0, & n &= \langle u_{ij} h_{ik} h_{jk} \rangle = E,\\
f &= \langle u_j h_{ji} h_{ikk} \rangle = E-F, & p &= \langle u_{ij} h_{ik} h_{kj} \rangle = F,\\
g &= \langle u_j h_{ki} h_{ijk} \rangle = 0, & q &= \langle u_{ij} h_{ki} h_{jk} \rangle = 0,\\
h &= \langle u_{ij} h_j h_{ikk} \rangle = -H, & r &= \langle u_{ij} h_{ki} h_{kj} \rangle = G,
\end{aligned}
$$

$$
\begin{aligned}
s &= \langle u_{ikk} h_{ij} h_j \rangle = H-F-E,\\
t &= \langle u_{ijk} h_{ij} h_k \rangle = H-E,\\
u &= \langle u_{ikk} h_{ji} h_j \rangle = 0,\\
v &= \langle u_{ijk} h_{ji} h_k \rangle = 0,\\
w &= \langle u_{ijk} h_{jk} h_i \rangle = -F,\\
x &= \langle u_{ijkk} h_i h_j \rangle = F+E-H.
\end{aligned}
$$

According to the location of the derivatives, the invariants fall into 6 classes: from a to b, c to g, h to m, n to r, s to w, and finally x. All terms which vanish because $u_{ii} = 0$ or $h_{ii} = 0$ have been omitted.

Similarly, we can list all the vectors formed with one u, two h's, and two derivatives. It includes 26 vectors such as, for example

$$\langle u_i h_k h_{kmm}\rangle, \quad \langle u_{ki} h_m h_{km}\rangle,$$
$$\langle u_i h_{km} h_{km}\rangle, \quad \langle u_{kim} h_k h_m\rangle.$$

Since an averaged vector must vanish in an isotropic field, the divergence of each such vector must also vanish. This gives 26 equations. For the above examples one finds:

$$d+b = 0, \quad s+p+j = 0,$$
$$2e = 0, \quad x+t+w = 0.$$

Several of these equations are redundant and, after reduction, only four quantities are independent. We selected the invariants h, n, p, and r and re-labelled them $-H$, E, F, and G. In terms of these new parameters, the 26 equations lead to the equalities listed above.

The Lorentz forces F_i are given by

$$F_i = h_k h_{ik} - h_k h_{ki}. \tag{A 1}$$

Their rate of work is $u_i F_i$ and simple operations show that

$$\langle u_i F_i\rangle = \langle u_i h_k h_{ik}\rangle = -\langle u_{ik} h_i h_k\rangle = -X. \tag{A 2}$$

We can always rotate the axes in such a way that, at a particular point, h_i lies along the x_1 axis. Then we have $u_i F_i = -\partial u_1/\partial x_1 h_1^2$, which shows that X measures the mean rate of stretching of h^2 produced by the velocity field.

The electric current density I_i is simply the curl of h_i. The quantity $u_{ik} I_i I_k$ measures the mean rate of stretching of I^2 by the velocity. The list given in this appendix leads to the following result:

$$\langle u_{ik} I_i I_k\rangle = F - E - G = Y. \tag{A 3}$$

We can also form the scalar product of the vorticity and the curl of the Lorentz forces F_i. Our list of invariants leads to the result

$$\langle \mathrm{curl}\, u_i \,.\, \mathrm{curl}\, F_i\rangle = E + F - H = \Omega. \tag{A 4}$$